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# A New High-Precision Mode Acceleration Method for Calculating Frequency Response of Non-Classically Damped Systems

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***Abstract*—The modal truncation problem of non-classically damped systems is constantly encountered in the dynamic analysis of engineering. The present study is designed to calculate the frequency response functions of the non-classically damping systems accurately on account of the Neumann expansion theory and the frequency shifting technique. Considering the first and the second term influence of the Neumann expansion equations in the frequency response analysis of the viscoelastic systems, we could correct the modal truncation problem of model displacement method. The property given in the study shows that this correcting method can reduce the high-order modes that can't be calculated to the lower-order modes that are easier to be computed. And the proposed method can also solve the problem causing by the singularity of stiffness matrix. The result of case given in the article shows that it can improve the accuracy of harmonic response effectively compared model displacement.**

***Keywords-harmonic response analysis; frequency shifting technique; model displacement method***

## I. INTRODUCTION

In many engineering problems, dynamic analysis, vibration control, structural design and damage detection is always an important part of it. Thus, the design of the algorithm and error control during the process to calculate the frequency response function plays an important role. The design needs to be quick and accurate for calculating the system frequency response function and has a great practical significance. With the amount of degrees of the modes considered in dynamic response analysis increasing, the process of computing all the frequencies response functions can be extraordinarily time consumption. However, in fact, the only modes considered in the frequencies response analysis are the modes located in the range of frequencies of interest. Unfortunately, since the method neglect the contribution of the higher-order modes and the viscoelastic modes, there will be some error existing in the modal truncation. Thus, many modified methods are proposed to solve the error accounting in the modal truncation. In recent decades, lots of studies have been done centered on the model reduction by using dimensionality reduction techniques in many research orientations.

Mode displacement method is the most basic method to solve statically indeterminate structures for its simple calculating process and accurate calculation results. In addition, model superposition methods also have an extensively use in structural field. Since 19th century, model reduction technique is hot spot in the computing of frequency response functions and the structural dynamics response analysis, the most common method is mode superposition method (MSM) that was presented by Rayleigh [1]. However, this method can have some improvements of the original MSM by using different vectors in the procedure of Neumann expansion [2]. Craig and Bampton [3] also gave a method to increase the accuracy by analyzing the nonlinear dynamic stability of an actual large-scale rotor-bearing system, which is called the fixed-interface reduction method for the fixing boundary of mode of system. Based on the free vibration modes and the available modes of the engineering structure, the mode displacement method [4] have been proposed by representing the displacement in a harmonic way. But this condition will not be always satisfied, so this kind of will not be suitable for the forced system. Mode acceleration method (MAM) is put forward to solve this problem by considering the superposition of the available modes and the free vibration modes.

Therefore, the MAM is a static correction method because of zero frequency. The experiments showed that mode acceleration method can really enhance the accuracy of the frequency response and simplify calculation of the FRF. But in the real situation, these constraint conditions will not be always satisfied. It's means that the error of modal truncation still exists. To solve this problem, many scholars struggle for it year after year, and have achieved gratifying successes. For example, Mario and Giuseppe [5] proposed a modified method for dynamic frequency response analysis of the systems in the reference. And the numerical applications are also showing that the proposed method can improve calculating efficiency. Certainly, some other corrections method can also have a good performance in improving the accuracy of dynamic response including dynamic correction method [6], high-precision modal superposition methods, self-adapting superposition method, correction continuous systems methods and so on.

With the widely use of non-viscous damping to analyze mechanical systems and dynamic frequency response calculating. The calculating of frequency response of non-viscously damped system has become increasingly important. To enhance the accuracy of the frequency response functions matrix, the articles proposed a method, which tries to estimate the influence of the modes that used to be irrespective and consider the nonviscously systems by taking the first one or two terms of Neumanm expansion into consideration. It's clear that with the number of modes used in the modal analysis of viscoelastic system increasing, the modal truncation error will accumulate gradually. A method is present to solve this problem by considering the lower mode and the firs terms' contribution in Neumann expansion. As to the non-proportionally systems, a method basing on the hybrid expansion is proposed to compute the response functions of the systems.

This study is devised to compute the harmonic responses of the available modes accurately. From the property obtained in the study on account of the Neumann expansion theorem and the frequency shifting technique, it's evident that the higher modes' frequency response function can express as equations consisting of the lower available modes and system matrices. We can use this property to simplify the higher modal truncation error. Certainly, we can use this method to improve the accuracy of frequency responses functions by dividing the frequency range into several sub-frequency ranges of interest and selecting different values for per sub-frequency range.

## II. BACKGROUND OF THEORY

The equations of motion for a linear non-viscously damped system with zero Initialization, obeys the governing equation

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t) \tag{1}$$

where $M$ , $C$ and $K \in R^{N\times N}$ are, the mass damping and stiffness matrices, $f(t)$ is the forcing vector. $u(t)$ is the displacement vector. In the sensitivity analysis of damped systems, $u(t)$ can also be called as the response vector.

In order to more aptly describe the phenomenon of "memory" of solid material or hysteresis effect, in 1874, Boltzmann put forward Boltzmann's superposition principle of linear viscoelastic materials. Later, in 1928, Volterra give the theory of hysteretic or memory in viscoelastic hereditary materials. So the damping force can be expressed as

$$f_d(t) = \int_0^t g(t-\tau)\dot{u}(t)d\tau \tag{2}$$

where $g(t)$ is a matrix of kernel function. Different places and areas have different choices of kernel function. Certainly, the theoretically how to choose kernel function remains unsolved. .

The equations of motion of a linear non-classically damped system with zero initial condition is

$$M\ddot{u}(t) + K_v\int_0^t g(t-\tau)\frac{\partial u(t)}{\partial \tau}dt + Ku(t) = f(t) \tag{3}$$

where $K_v$ which is the damping coefficient matrix. $g(t)$ is a kernel function that has different names in the different places.

If the loading function is harmonic, that is $f(t) = F_h(s)\exp(st)$ with $s = iw$ and $F_h \in R^N$ , the steady-state frequency response will also be harmonic, i.e. $u(t) = U_h(s)\exp(st)$. Taking the places of $u(t)$ and $f(t)$ in (3), we can obtain

$$(s^2M + sG(s) + K)U_h(s) = F_h \quad \text{ορ} \quad D(s)U_h(s) = F_h \tag{4}$$

Here $G(s) = K_V L[g(t)]$ and $L[]$ denotes the Laplace transform, we know that $G(s)$ can also be expressed as

$$G(s) = \sum_{k=1}^{n} \frac{c\mu_k}{s+\mu_k} K_v \tag{5}$$

Here $c$ and $\mu_k$ are the relaxation parameters. And for the dynamic stiffness matrix, it can be expressed as

$$D(s) = s^2M + sG(s) + K \tag{6}$$

The accurate steady-state frequency response that we want to get can be acquired by utilizing the direct frequency response method. For the characteristic equation

$$\det\left|s^2M + sG(s) + K\right| = 0 \tag{7}$$

The eigenvalue $\lambda_j$ are the roots of it. And where $\varphi_j$ denotes the $j$ th eigenvector and can be rewritten in another way

$$({\lambda_j}^2 M + \lambda_j G(\lambda_j) + K)\varphi_j = 0 \tag{8}$$

In addition, asymmetric-matrices problem may also arise for using the state-space approaches. However, these normal modes based on those approaches still have some error when computing the frequency response functions particularly for high-dimensionality damped systems. Furthermore, we can avoid the convergence problem by considering iterative strategy.

The complex FRF matrix and the response vector $U_h$ can be obtained by

$$H(s) = \sum_{j=1}^{m}\frac{\varphi_j{\varphi_j}^T}{\theta_j(s-\lambda_j)}, \quad U_h(s) = \sum_{j=1}^{m}\frac{{\varphi_j}^T F_h \varphi_j}{\theta_j(s-\lambda_j)} \tag{9}$$

where $\theta_j = \varphi_j{}^T \frac{\partial D(s)}{\partial s}\Big|_{s=\lambda_j} \varphi_j$

and $\frac{\partial D(s)}{\partial s}\Big|_{s=\lambda_j} = 2\lambda_j M + G(s) + \lambda_j \frac{\partial D(s)}{\partial s}\Big|_{s=\lambda_j}$

And this situation is suitable for the ideal situation that eigenvalues are separated or non-repeated. For the complexity of the non-viscously damped systems, the model will be presented by a large number of different equations. It's means that the modal-truncation error still exists.

To solve this problem, we introduce the modal truncation error. Given that the frequency from $L_1$ th to $L_2$ th of interest can be computed, the error of modal truncation of the modal displacement method can be obtained by

$$E_{MDM}(s) = \sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{(s-\lambda_j)\theta_j} + \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{(s-\lambda_j)\theta_j} \tag{10}$$

For the inverse matrix, Neumann expansion can be expressed in the following way

$$(I_N - A)^{-1} = I_N + A + A^2 + A^3 + \cdots \tag{11}$$

Here $A \in R^{N\times N}$ and $I$ on behalf of the unit matrix. Given that (9) meets converge condition, the power-series expansion can tend to the exact result. The FRF matrix presented can be rewritten into the matrix form as

$$H(s) = -U\Theta^{-1}(\Lambda - sI_m)^{-1}U^T \tag{12}$$

where $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_m]$ , $U = [\varphi_1, \varphi_2, \ldots, \varphi_m]$ and $\Theta = \mathrm{diag}[\theta_1, \theta_2, \ldots, \theta_m]$.

Let $\bar{s} = s - \mu$ and using the Neumann expansion, the frequency response function matrix can be expressed by

$$H(s) = -\sum_{r=1}^{\infty} U\Theta^{-1}\bar{s}^{r-1}(\Lambda - \mu I_m)^{-r}U^T \tag{13}$$

where $\mu$ is a complex frequency shift constant.

The dynamic stiffness matrix $D(s)$ given in (6), can also be expressed as

$$\begin{aligned} D(s) &= (s-\mu)^2 M + (s-\mu)(G(s) + 2\mu M) \\ &\quad + (K + \mu G(s) + \mu^2 M) \end{aligned} \tag{14}$$

Let

$$\begin{aligned} \bar{K}(s) &= K + \mu G(s) + \mu^2 M \\ \bar{G}(s) &= G(s) + 2\mu M \end{aligned} \tag{15}$$

Comparing (12) and (13), let $\bar{s} \to 0 (s \to \mu)$, we can get

$$\begin{aligned} U\Theta^{-1}(\Lambda - \mu I_m)^{-1}U^T &= \lim_{s\to\mu} \bar{K}(s)^{-1} \\ &= (K + \mu G(\mu) + \mu^2 M)^{-1} \end{aligned} \tag{16}$$

## III. A METHOD TO IMPROVE THE ACCURACY OF MODE ACCELERATION METHOD

Based on the free vibration modes and available of the structure, the mode displacement method have been presented by using a time-harmonic representation for the displacement of the unforced system. But this condition will not be always satisfied, so this kind of will not be suitable for the forced system. Mode acceleration method (MAM) is proposed to reduce the modal truncation error by considering the effect of higher modes. From the (16), we can see that this problem of singular problem of stiffness matrix have been overcome while incoming the frequency shift constant $\mu$.

Substituting $\bar{K}(s)$ and $\bar{G}(s)$ in (15), the equation can be rewritten in the anther way as

$$\begin{aligned} U\Theta^{-1}(\Lambda - \mu I_m)^{-2}U^T &= (K + \mu G(\mu) + \mu^2 M)^{-1} \\ &\cdot (G(\mu) + 2\mu M) \cdot (K + \mu G(\mu) + \mu^2 M)^{-1} \end{aligned} \tag{17}$$

By using the Neumann expansion theorem and let $\bar{s} = s - \mu$, the FRF matrix given in (13) can be alternatively expressed as

$$H(s) = -\sum_{r=1}^{\infty}\sum_{j=1}^{m} \frac{\bar{s}^{r-1}\varphi_j \varphi_j{}^T}{\theta_j(\lambda_j - \mu)} \tag{18}$$

When $r = 1, 2$, considering the contribution of the first and second term of the Neumann expansion of the higher modes, (18) can be presented in the following way by utilizing the lower available modes

$$\begin{aligned} H_1(s) &= -\sum_{j=1}^{m} \frac{\varphi_j \varphi_j{}^T}{\theta_j(\lambda_j - \mu)} = (K + \mu G(\mu) + \mu^2 M)^{-1} \\ H_2(s) &= -\sum_{j=1}^{m} \frac{\bar{s}\varphi_j \varphi_j{}^T}{\theta_j(\lambda_j - \mu)^2} = (K + \mu G(\mu) + \mu^2 M)^{-1} \\ &\quad \cdot (G(\mu) + 2\mu M) \cdot (K + \mu G(\mu) + \mu^2 M)^{-1} \end{aligned} \tag{19}$$

Assuming the frequency range from $L_1$ th to $L_2$ th of interest can be calculated, the response can be computed precisely in the following way

$$U_h(s) = \sum_{j=L_1}^{L_2} \frac{\varphi_j{}^T F_h \varphi_j}{(s-\lambda_j)\theta_j} + E_{MDM}(s) \tag{20}$$

The same process as (13), using Neumann expansion, we can obtain

$$\begin{aligned} E_{GMAM}(s) = -\sum_{k=0}^{\infty}(s-\mu)^k \\ \cdot\left[\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^{k+1}} + \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^{k+1}}\right] \end{aligned} \tag{21}$$

For the (21), it considers the first term of the right-hand equation. When $k=0$, the equation can be expressed as

$$E_{MDM}(s) \approx -\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)} - \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)} \tag{22}$$

Then give a frequency shift value $\mu$ to (22), we can obtain

$$\begin{aligned} \sum_{j=L+1}^{m} \frac{\varphi_j \varphi_j{}^T}{(\lambda_j-\mu)^2\theta_j} = (K+\mu G(\mu)+\mu^2 M)^{-1} \\ \cdot(G(\mu)+2\mu M)\cdot(K+\mu G(\mu)+\mu^2 M)^{-1} - \sum_{j=1}^{L} \frac{\varphi_j \varphi_j{}^T}{(\lambda_j-\mu)^2\theta_j} \end{aligned} \tag{23}$$

We can see that (23) can reduce the high modes to the lower modes. That is to say, we can use this equation to implement dimensionlity reduction. When $k=1$, the above equation will be rewritten in this way

$$\begin{aligned} E_{MDM}(s) \approx -\left[\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)} + \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)}\right] \\ -(s-\mu)\left[\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^2} + \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^2}\right] \end{aligned} \tag{24}$$

In order to compute efficiently, we remark the first term $E_1$ and the second term $E_2$. That is

$$\begin{aligned} E_1 &= -\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)} - \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)} \\ E_2 &= -(s-\mu)\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^2} - (s-\mu)\sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^2} \end{aligned} \tag{25}$$

In the equation, all parts can be computed. By considering the influence of the second term of the truncation error of model displacement method, it's no doubt that the error can be reduced in this way. For the singularity of stiffness matrix, the results in the number of terms that we can use are merely the first and the second one. Thus, the response in (9) can be calculated can be expressed by

$$\begin{aligned} U_h(s) &= \sum_{j=L_1}^{L_2} \frac{\varphi_j{}^T F_h \varphi_j}{(s-\lambda_j)\theta_j} + E_{MDM}(s) \\ &= \sum_{j=L_1}^{L_2} \frac{\varphi_j{}^T F_h \varphi_j}{(s-\lambda_j)\theta_j} + E_1 + E_2 + E_e \end{aligned} \tag{26}$$

Using the upper bound of the $r$ th component of the error vector $E_e$ , we can obtain the below equation

$$\left|E_e\right|_r \le \left|\sum_{k=2}^{\infty} \bar{s}^k \left[\sum_{j=1}^{L_1-1} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^{k+1}} + \sum_{j=L_2+1}^{m} \frac{\varphi_j{}^T F_h \varphi_j}{\theta_j(\lambda_j-\mu)^{k+1}}\right]\right|_r \tag{27}$$

Here $\left|\varphi_j\right|_r$ represents the $r$ th element of the vector. Comparing to the error of generalized mode acceleration method, we can know that the error is becoming smaller by considering the influence of the second term of the Neumann expansion of the model displacement error.

## IV. EXAMPLE AND DISCUSS

In this present part, one case of the harmonic forced vibration of non-classically damped system is shown, which is a four DOF nonviscously system with free-free boundary condition [11].

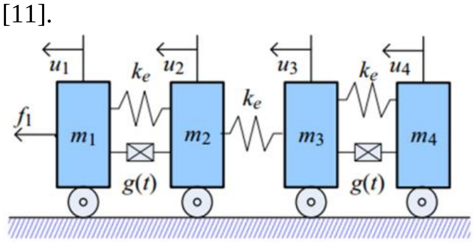


Figure 1. Four DOF non-classically damped system with free-free boundary condition

The system matrices of the nonviscously damped mode, shown in **Fig .1**, are $M$ , $K$ and $G$ . It's obvious that the energy dissipation is not uniformly distributed in the whole system. That is say, the system is a non-classically damped system.

Suppose the interesting frequency range is 12-28 rad/s. According to the present theorem in [11], the frequency shift value is $\mu = 20i$. Four elastic modes are covered in the frequency range of interest.

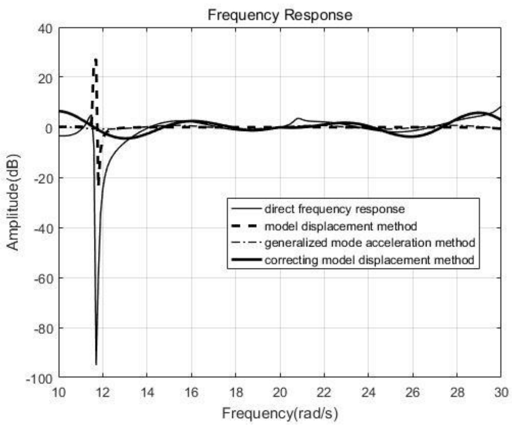


Figure 2. The FRF of the mode in the frequency range 10-30 rad/s

**Fig .2** shows that the FRF of the second DOF in mode over the frequency range of interest. Since the frequency shift value $\mu = 20i$, it's evident that the modal truncation error caused by the model displacement method can be reduced when the considering frequency is located in the frequency range of interest. For example, in 14-30 rad/s, the correcting method presenting in this study can have a better accuracy that the generalized acceleration method proposed by Li et al. in [11]. That is say the results have a better performance when the frequency tends to the frequency shift $\mu$.

## V. Conclusions

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