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Equivalence on Quadratic Lyapunov Function Based Algorithms in Stochastic Networks

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***Abstract*—Quadratic Lyapunov function based Algorithms (QLAs) for stochastic network optimization problems, which are cross-layer scheduling algorithms designed by Lyapunov optimization technique, have been widely used and studied. In this paper, we investigate the performance of using Lyapunov drift and perturbation in QLAs. By analyzing attraction points and utility performance of four variants of OQLA (Original QLA), we examine the rationality of OQLA for using the first-order part of an upper bound of Lyapunov drift of a function L_1. It is proved that either using the real Lyapunov function (L_2) of networks under QLA or using the entire expression of Lyapunov drift does not improve backlog-utility performance. The linear relationship between the attraction point of backlog and perturbation in the queue is found. Simulations verify the results above.**

Keywords-Component; Lyapunov optimization; QLA; Lyapunov function; Backlog-utility performanc; Stochastic network optimization

I. INTRODUCTION

Lyapunov optimization technique is an effective method to design online cross-layer scheduling algorithms for stochastic network. The Lyapunov optimization technique is able to stabilize the network while achieving close-to-optimal utility performance [1][2]. Among the multiple advantages of using the Lyapunov optimization technique in stochastic network optimization, the most significant one is that probability distributions in the network are not necessarily known but able to be obtained by the Lyapunov optimization technique, adapt to networks with any distributions. The Lyapunov optimization technique has been used in various scenarios, including wireless communication networks [3][4], energy harvesting networks [5], processing networks [6], and even financial systems [7].

We mainly focus on the Quadratic Lyapunov function based Algorithm (QLA). For clarity, the original QLA proposed in e.g. [3] is referred as OQLA henceforth. OQLA is designed to greedily minimize an expression consisting of two parts, one of which is the first-order part of the upper bound of Lyapunov drift of a specific Lyapunov function (this function is denoted as L_1), e.g. [2]. However, using the entire expression of Lyapunov drift when minimizing seems to perform better than using first-order part of upper bound of Lyapunov drift. Moreover, L_1 used in design OQLA is apparently not the actual Lyapunov function of networks under OQLA. Because if L_1 is the actual Lyapunov function of networks under OQLA, queue backlog should be attracted by zero. But both analysis (e.g. in [8]) and simulation (e.g. in [2]) show that queue length is attracted by a non-zero value in networks under OQLA. It seems quite arbitrary to design OQLAs by using L_1 and the first-order part. To our best knowledge, no comparisons about the delay-utility performance between using the first-order part and the entire expression and between using the real Lyapunov function (L_2) and L_1 are given in the previous works of QLA. There are several works focusing on reducing the backlog of OQLA, e.g. [9][10].

We now summarize the main contributions of this paper in the following. 1) We prove that either using the entire Lyapunov drift expression instead of the first-order part of upper bound of Lyapunov drift of L_1 or using the real Lyapunov function instead of L_1 doesn't improve utility and delay performance. 2) We demonstrate and prove the utility and delay performance of QLA with Perturbed Data Queues (QLA-PDQ-P & QLA-PDQ-E) and QLA based on Entire expression of Lyapunov drift (QLA-E).

The rest of the paper is organized as follows: In Section II, we state our network model. After some preliminary information is given in Section III, we present our main results in Section IV, including a Lyapunov function and backlog-utility performance of variants of OQLA. Explanations of those results are given to show the rationality of OQLA and the relationship between attraction point and perturbation. In Section V, we prove results in Section IV. Section VI provides simulation results of QLA-PDQ-E, QLA-PDQ-P. We conclude in Section VII.

II. NETWORK MODEL

In this section we specify the network model we use, which is also widely used in other works about QLA.

A. Network States & Actions

The network consists of r ($r \in \mathbb{Z}^+$ and is finite) queues. There are M network states forming the state set $\mathcal{S}$. Each state is denoted as s_i, indicating the current network parameters. The network operates in slotted time.

Denote network state at time t as $s(t)$. We assume that $s(t)$ is stationary ergodic processes with finite state space and evolves according to some general probability law, under which there exists a steady state distribution of $s(t)$. Let p_{s_i} denote its steady state probability of being in state s_i, i.e,$p_{s_i} = \Pr\{s(t) = s_i\}$. At each timeslot t when the state $s(t) = s_i$, the network controller chooses an action $\alpha(t)$ from a set $\mathcal{A}^{s_i}$, i.e. $\alpha(t) = \alpha^{s_i}$ for some $\alpha^{s_i} \in \mathcal{A}^{s_i}$. The set $\mathcal{A}^{s_i}$ is called the feasible action set for network state s_i and is assumed to be time-invariant and compact for all $s_i \in \mathcal{S}$. Denote the action vector $< \alpha^{s_1}, \dots, \alpha^{s_M} >$ as $\boldsymbol{\alpha}$.

B. Queues

Let $\mathrm{Q(t)} =< \mathrm{Q_1(t)}, \dots, \mathrm{Q_r(t)} >$ denote the data queue backlog vector process of the network, where $\mathrm{Q_i(t)}$ is non-negative. Each queue is updated in the following way.$\mathrm{Q_i(t+1)} = \max\{\mathrm{Q_i(t)} - \mathrm{b_i(t)}, 0\} + \mathrm{A_i(t)}$. Queue i is mean rate stable (shorted for "stable" hereafter) means:$\lim_{t\to\infty} \frac{\mathbb{E}\{|Q_i(t)|\}}{t} = 0$[2]. The network is stable if all data queues are stable. Virtual queues can be used to represent time-averaged constraints. Simulations in Section VI consider a problem with time-averaged constraints to show our analysis and conclusions still hold.

Lyapunov function used in OQLA is defined as $L_1(t) = \frac{1}{2}\{\sum_{i=1}^{r} Q_i\,(t)^2\}$. Lyapunov drift of $L_1(t)$ is $D_1(t) = L_1(t+1) - L_1(t)$. The first-order part of upper bound of $D_1(t)$ used in OQLA is denoted as Δ_1.

Define Lyapunov function $L'(t)$ as $L'(t) = L'(t, \mathbf{C}) = \frac{1}{2}\{\sum_{i=1}^{r}(\,Q_i(t) - C_i)^2\}$, where $\mathbf{C} =< C_1(t), \dots, C_r(t) >$ denotes the perturbation of data queues. Lyapunov drift of the$L'(t)$ is $D'(t, \mathbf{C}) = L'(t+1) - L'(t)$. The first-order part of upper bound of $D'(t)$ is denoted as $\Delta'_{\mathbf{C}}$. Note that $L_1(t) = L'(t, \mathbf{0})$. Define Lyapunov function $L_2(t)$ as $L_2(t) = L'(t, \boldsymbol{\gamma}^*(V)_{\mathrm{OQLA}})$, Lyapunov drift of which is denoted as $D_2(t)$. The first-order part of upper bound of $D_2(t)$ is denoted as Δ_2. Expressions of $D_1(t)$, Δ_1, $D_2(t)$, Δ_2, $D'(t)$ and Δ' are shown and proved in [11].

C. Stochastic Optimization Problem

We consider a stochastic optimization problem with utility maximization. A utility function f is a function of network parameters, such as total throughput or energy-cost. Define time-average expectation of f(t) over the first T timeslots as $\bar{\mathrm{f}} = \frac{1}{\mathrm{T}}\sum_{\tau=1}^{\mathrm{T}} \mathbb{E}\,\{\mathrm{f}(\tau)\}$.

A network controller is designed to solve this problem, which operates a network with the goal of minimizing $\limsup_{t\to\infty}\overline{f}$, subject to the queue stability and additional time-average constraints. The case maximizing f can be treated the same way by letting $f' = -f$. We assume the network controller can observe $s(t)$ at the beginning of every timeslot t, but the p_{s_i} probabilities are not necessarily known. Thus f is can be regarded as a function of $s(t)$ and $\alpha(t)$, i.e. $f(t) = f(s(t), \alpha(t))$. Define f^{opt} as the maximum value of $\limsup_{t\to\infty}\overline{f}$ over all control policies that satisfies the stability and time-average constraints.

This problem is solved by OQLA in the way that each timeslot the network controller chooses the action that greedily minimizes $\Delta_1 + Vf$, where V is a positive constant $V \geq 1$ [2].

OQLA: $\alpha(t) = \mathrm{argmin}_{\alpha(t)\in\mathcal{A}^{s(t)}}\{\Delta_1(t) + Vf(t)\}$

It has been proved that OQLA stabilizes the network and achieves maximum utility asymptotically [2].

The following four variants of OQLA are analyzed in this paper.

- QLA-P: QLA using first-order Part of upper bound of Lyapunov drift of L_2 with a parameter V:
  $\alpha(t) = \mathrm{argmin}_{\alpha(t)\in\mathcal{A}^{s(t)}}\{\Delta_2(t) + Vf(t)\}$
- QLA-E: QLA using Entire expression of Lyapunov drift of L_1 with a parameter V:
  $\alpha(t) = \mathrm{argmin}_{\alpha(t)\in\mathcal{A}^{s(t)}}\{D_1(t) + Vf(t)\}$
- QLA-PDQ-P: QLA with Perturbed Data Queue using first-order Part of upper bound of Lyapunov drift of L' with parameters V and $\mathbf{C}$:
  $\alpha(t) = \mathrm{argmin}_{\alpha(t)\in\mathcal{A}^{s(t)}}\{\Delta'(t, \mathbf{C}) + Vf(t)\}$
- QLA-PDQ-E: QLA with Perturbed Data Queue using Entire expression of Lyapunov drift of L' with parameters V and $\mathbf{C}$:
  $\alpha(t) = \mathrm{argmin}_{\alpha(t)\in\mathcal{A}^{s(t)}}\{D'(t, \mathbf{C}) + Vf(t)\}$

Note that OQLA is equivalent to QLA-PDQ-P when $\mathrm{C} = 0$, QLA-E is equivalent to QLA-PDQ-E when $\mathrm{C} = 0$, QLA-P is equivalent to QLA-PDQ-P when $\mathrm{C} = \gamma^*(\mathrm{V})_{\mathrm{OQLA}}$.

D. An Example of the Model

Here we provide an example to illustrate our model, which will be used in Section VI. There are 3 nodes in the network. Each node can communicate with another. Thus there are 6 queues in the network. Denote the queue from node i to node j as (i, j) ($\mathrm{i} \neq \mathrm{j}$), backlog of which is denoted as $\mathrm{Q_{ij}(t)}$. Possible data flows of queues are shown in Table [t_1].

TABLE I. QUEUES AND DATA FLOWS IN A NETWORK OF 3 NODES

Node No.	Queue No.	Possible Input		Possible Output	
N1	Q(1,2)	E[a]	Q(3,2)	N2	Q(3,2)
	Q(1,3)	E	Q(2,3)	N3	Q(2,3)
N2	Q(2,1)	E	Q(3,1)	N1	Q(3,1)
	Q(2,3)	E	Q(1,3)	N3	Q(1,3)
N3	Q(3,1)	E	Q(2,1)	N1	Q(2,1)
	Q(3,2)	E	Q(1,2)	N2	Q(1,2)

a. "E" is short for exogenous arrivals from outside the network

Network state consists of link state l_{ij} of from node i to node j, where $l_{ij} \in \{l_k, k = 1,2,3,4\}$. $l_k(k = 1,2,3,4)$ denote link status Good, Common, Bad and Disconnected respectively. Define $\xi_{ij} \in \{3,2,1,0\}$, where $\xi_{ij} = 3,2,1,0$ if $l_{ij} = l_1, l_2, l_3, l_4$ respectively. Link state is i.i.d. and $l_{ij} = l_k(k = 1,2,3,4)$ with equal probabilities. There are totally 4^6 network states.

Exogenous arrival into queue (i,j) from outside the network is denoted as a_{ij}. Maximum exogenous arrival to a queue is $A_{\max} = 6$, which means a_{ij} satisfies the constraint $0 \leq a_{ij} \leq A_{\max}$. Service allocated from queue (i,j) to node k is denoted as b_{ijk}. Power-service function is defined as $b_{ijk} = \ln\{1 + \xi_{ik} p_{ijk}\}$. Packets from queue (i,j) can only be transmitted to either its destination node j, or queue (k,j) of the other node k, which means $p_{iji} = 0$. Maximum power allocated to a queue is $P_{\max} = 6$, which means p_{ijk} satisfies $0 \leq p_{ijk} \leq P_{\max}$. Define the power out of node i as $p_i^N(t) = \sum_{j\neq i, k\neq i} p_{ijk}(t)$. Time-average power out of any node should be lower than P_{av}, i.e. $\limsup_{t\to\infty} \frac{1}{t}\sum_1^t p_i^N(t) \leq P_{\text{av}}$. Thus the corresponding virtual queue X_i updates according to $X_i(t+1) = \max\{X_i(t) + p_i^N(t) - P_{\text{av}}, 0\}$. There are totally 3 virtual queues. Utility function is defined as $f(t) = \sum_{i,j} \ln(1 + a_{ij})$ to represent total throughput of the network.

Each timeslot, according to maximum constraints of a_{ij}'s and p_{ijk}'s and time-average constraints of p_i^N network controller decides the amount of packets into each queue and decides the power allocated to each queue, i.e. network controller decides the values of a_{ij}'s and p_{ijk}'s.

III. PRELIMINARY

A. Definitions

Let $\|\mathrm{Y}\|$ denote the inner product of Y, i.e. $\|\mathrm{Y}\| = \mathrm{Y} \bullet \mathrm{Y}^{\mathrm{T}}$.

Definition 1 (Attraction Point). Define the attraction point of a stochastic process Y(t) as follows.

Y^* is the Attraction Point of a process Y(t) with parameters ν, D and η if: There exist $\nu > 0$, $\mathrm{D} > \eta > 0$, such that $\mathrm{D} = \mathrm{D}(\nu)$, $\eta = \eta(\nu)$, and whenever $\|\mathrm{Y}(t) - \mathrm{Y}^*\| \geq \mathrm{D}$, we have $\mathbb{E}\{\|\mathrm{Y}(t + \mathrm{T}_\nu) - \mathrm{Y}^*\| - \|\mathrm{Y}(t) - \mathrm{Y}^*\| | \mathrm{Y}(t)\} \leq -\eta$.

Definition 2 (Locally Polyhedral). Define locally polyhedral property same as in [9].

Definition 3. Define functions $\mathrm{l}_{k,s_i}, \mathrm{g}_{k,s_i}, \mathrm{l}_k, \mathrm{g}_k (k = 0, \dots, 4)$ as follows.

$$\mathrm{l}_{0,s_i}(\mathrm{Q}, \alpha^{s_i}, \mathrm{V}) = \{\mathrm{Vf}(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} \mathrm{Q}_i (\mathrm{A}_i(s_i, \alpha^{s_i}) - \mathrm{b}_i(s_i, \alpha^{s_i}))\}$$

$$\mathrm{l}_{1,s_i}(\mathrm{Q}, \alpha^{s_i}, \mathrm{V}) = \{\mathrm{Vf}(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} (\mathrm{Q}_i - \gamma^*(\mathrm{V})_{\mathrm{OQLA}}) \times (\mathrm{A}_i(s_i, \alpha^{s_i}) - \mathrm{b}_i(s_i, \alpha^{s_i}))\}$$

$$\mathrm{l}_{2,s_i}(\mathrm{Q}, \alpha^{s_i}, \mathrm{V}) = \{\mathrm{Vf}(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} (\frac{1}{2}(\mathrm{A}_i(s_i, \alpha^{s_i}) - \tilde{\mathrm{b}}_i(s_i, \alpha^{s_i}))^2 + \mathrm{Q}_i(\mathrm{A}_i(s_i, \alpha^{s_i}) - \tilde{\mathrm{b}}_i(s_i, \alpha^{s_i})))\}$$

$$\mathrm{l}_{3,s_i}(\mathrm{Q}, \alpha^{s_i}, \mathrm{V}, \mathrm{C}) = \{\mathrm{Vf}(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} (\mathrm{Q}_i - \mathrm{C}_i)(\mathrm{A}_i(s_i, \alpha^{s_i}) - \mathrm{b}_i(s_i, \alpha^{s_i}))\}$$

$$\mathrm{l}_{4,s_i}(\mathrm{Q}, \alpha^{s_i}, \mathrm{V}, \mathrm{C}) = \{\mathrm{Vf}(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} (\frac{1}{2}(\mathrm{A}_i(s_i, \alpha^{s_i}) - \tilde{\mathrm{b}}_i(s_i, \alpha^{s_i}))^2 + (\mathrm{Q}_i - \mathrm{C}_i)(\mathrm{A}_i(s_i, \alpha^{s_i}) - \tilde{\mathrm{b}}_i(s_i, \alpha^{s_i})))\}$$

Define g_{k,s_i} as $\inf_{\alpha^{s_i} \in \mathcal{A}^{s_i}} \mathrm{l}_{k,s_i}(\mathrm{Q}, \alpha^{s_i})$. Define $\mathrm{g}_k(\mathrm{Q})$ as $\sum_{s_i} \mathrm{p}_{s_i} \mathrm{g}_{k,s_i}$. Define $\mathrm{l}_k(\mathrm{Q}, \alpha)$ as $\sum_{s_i} \mathrm{p}_{s_i} \mathrm{l}_{k,s_i}(\mathrm{Q}, \alpha^{s_i})$. Note that $\mathrm{g}_k = \inf_{\alpha} \mathrm{l}_k(\mathrm{Q}, \alpha^{s_i})$.

B. Assumptions

We list the assumptions used hereafter, which are as same as the ones in [8]. These assumptions hold in many network utility optimization problems and aren't so rigorous as they appear. Explanations can be found in [11].

Assumption 1. Local maximum point of $g_i(i = 0, \dots, 4)$ is unique on $\mathbb{R}^r$, denoted as $\boldsymbol{\gamma}_i^*(V)$.

Assumption 2. $g_i(i = 0, \dots, 4)$ is locally polyhedral at its maximum point.

Assumption 3. ϵ-slackness[9] holds for the network.

Assumptions 1 and 2 are about the property of utility function, while Assumption 3 is about the network.

C. Lemmas

Before moving further we introduce the following three lemmas, proofs of which are omitted for brevity and can be found in [2][3][8].

Lemma 1. $\boldsymbol{\gamma}^* \geq \mathbf{0}$ is the unique attraction point of $\mathbf{Q}(t)$ if:

1) A function $g(\mathbf{Q})$ is locally polyhedral at $\boldsymbol{\gamma}^*$.
2) For all s_i, there exists a positive constant C satisfying

$$\|\mathbf{Q}^{s_i}(t+1) - \boldsymbol{\gamma}^*\|^2 - \|\mathbf{Q}(t) - \boldsymbol{\gamma}^*\|^2 \leq C - 2(g_{s_i}(\boldsymbol{\gamma}^*) - g_{s_i}(\mathbf{Q}(t)))$$

Lemma 2. For network under OQLA with a parameter V, for any point $\mathbf{Q}$, we have

$$g_0(\mathbf{Q}, V) \leq g_0(\boldsymbol{\gamma}_0^*(V), V) \leq V f^{\text{opt}}$$

IV. MAIN RESULTS

We summarize analysis results here.

Denote the Euclid ball centered at $\mathbf{Q}_0$ with radius r as $\mathbf{B}(\mathbf{Q}_0, r)$.

Theorem 1. When the network state is i.i.d., in the sense of conditional expectation, one Lyapunov function of the network under OQLA is

$$L_3(\mathbf{Q}) = \begin{cases} L_2 & \mathbf{Q} \notin \mathbf{B}(\boldsymbol{\gamma}_0^*(V), D_0) \\ 0 & \mathbf{Q} \in \mathbf{B}(\boldsymbol{\gamma}_0^*(V), D_0) \end{cases}$$

Note that L_2 equals L_3 in most cases thus L_2 is used in QLA-P instead of L_3.

Theorem 2. For the network under QLA-P (QLA using first-order Part of upper bound of Lyapunov drift of L_2) with a parameter V, the following properties hold. The attraction point of queue backlog is $2\boldsymbol{\gamma}_0^*(V)$ Utility function satisfies $\limsup_{t\to\infty} \overline{f} \leq f^{\text{opt}} + B_1/V$ where B_1 is a positive constant.

Theorem 2 shows that using L_2 instead of L_1 when designing QLA, queue backlog is doubled while utility performance remains $\mathcal{O}(1/V)$ compared to OQLA. Thus using the Lyapunov function of the network under OQLA does not help the queue backlog and utility performance, only causing backlog to be larger.

Theorem 3. For the network under QLA-E (QLA using Entire expression of Lyapunov drift of L_1) with a parameter V, the following properties hold. Queue backlog is attracted by $\boldsymbol{\gamma}_2^*(V)$. $\boldsymbol{\gamma}_2^*$ approximately equals $\boldsymbol{\gamma}_0^*$ when V is large enough. Utility function satisfies $\limsup_{t\to\infty}\overline{f} \le f^{\mathrm{opt}} + B_2/V$, where B_2 is a positive constant.

Theorem 3 shows that using the entire Lyapunov drift expression instead of the first-order part of the upper bound of Lyapunov drift of L_1 when designing QLA, queue backlog and utility performance is not improved compared to OQLA. However, when using the entire expression, some problems with variables coupled loosely which can be solved in a distributed manner become problems with variables coupled tightly which can only be solved in a centralized manner. Thus using the entire expression increases the complexity. Therefore using the first-order part when designing QLA (such as in [2]) is reasonable.

Theorem 4. For the network under QLA-PDQ-P (QLA with Perturbed Data Queue using first-order Part of upper bound of Lyapunov drift of L') with parameters V and $\mathbf{C}$, the following properties hold. If $\boldsymbol{\gamma}_0^*(V) \ge -\mathbf{C}$, the attraction point of queue backlog $\boldsymbol{\gamma}_3^*$ equals $\boldsymbol{\gamma}_0^*(V) + \mathbf{C}$. Utility function satisfies $\limsup_{t\to\infty}\overline{f} \le f^{\mathrm{opt}} + \frac{B_3}{V}$, where B_3 is a positive constant.

Theorem 4 shows that using a positive $\mathbf{C}$ increases backlog while using a negative $\mathbf{C}$ decreases backlog. This idea is used when designing QLA-VPDQ.

Theorem 5. For the network under QLA-PDQ-E (QLA with Perturbed Data Queue using Entire expression of Lyapunov drift of L') with parameters V and $\mathbf{C}$, the following properties hold. The attraction point of queue backlog $\boldsymbol{\gamma}_4^*$ equals $\boldsymbol{\gamma}_2^*(V) + \mathbf{C}$, Utility function satisfies $\limsup_{t\to\infty}\overline{f} \le f^{\mathrm{opt}} + \frac{B_4}{V}$, where B_4 is a positive constant.

Theorem 5 shows that using the entire expression doesn't help to enhance queue backlog and utility performance even for QLA with perturbed data queue.

V. Proofs

A. *Proof of Theorem 1*

It can be seen from the definition of L_3 that $L_3 \ge 0$ and $L_3 = 0$ only when $Q \in B(\gamma_0^*(V), D_0)$. Thus we have from Lemma 1 that $\mathbb{E}\{\|Y(t+1) - Y^*\| - \|Y(t) - Y^*\| | Y(t)\} < 0$ when $Q \notin B(\gamma_0^*(V), D_0)$.

Thus L_3 is a Lyapunov function of the network in the sense of conditional expectation with the stability point $\mathbf{B}(\boldsymbol{\gamma}_0^*(V), D_0)$.

B. *Proof of Theorem 3*

Lemma 3. For the network under QLA-E, the following two equations hold.

For any Q_1 and Q_2, we have

$$g_2(Q_1) \le g_2(Q_2) + (Q_1 - Q_2) \bullet (A(t) - \tilde{b}(t)) \quad (1)$$

For any Q, we have

$$\| Q^{s_i}(t+1) - \gamma_2^* \|^2 - \| Q(t) - \gamma_2^* \|^2 \le C_2 - 2(\gamma_2^* - Q(t)) \bullet (A(t) - \tilde{b}(t)) \quad (2)$$

Proof of Lemma 3 can be found in [11].

1) *Property of Queue Backlog*

Attraction Point Property

From (1) and (2) we have for any Q

$$\| Q^{s_i}(t+1) - \gamma_2^* \|^2 - \| Q(t) - \gamma_2^* \|^2 \le C_2 - 2(g_{2,s_i}(\gamma_2^*) - g_{2,s_i}(Q(t))$$

Using Lemma 1 and noting that g_2 is polyhedral it can be concluded that γ_2^* is the unique attraction point of queue backlog Q(t).

Linear Property of Attraction Point

From the expression of g_2 we see that,

$$g_2(Q)/V = \inf_{\alpha} \sum_{s_i} p_{s_i} \{f(s_i, \alpha^{s_i}) + \sum_{i=1}^{r} (\frac{1}{2V}(A_i(s_i, \alpha^{s_i}) - \tilde{b}_i(s_i, \alpha^{s_i}))^2 + Q'_i(A_i(s_i, \alpha^{s_i}) - \tilde{b}_i(s_i, \alpha^{s_i})))\}$$

where $Q'_i = \frac{Q_i}{V}$. When V is large enough and $\frac{1}{2V}(A_i(s_i, \alpha^{s_i}) - \tilde{b}_i(s_i, \alpha^{s_i}))^2$ is small enough to be ignored, the right hand side is $g_0(Q', 1)$ which maximized at $\gamma_0^*(1)$. Therefore we have $\gamma_2^*(V) \approx V\gamma_0^*(1) = \gamma_0^*(V)$.

2) *Property of Utility Function*

By [11], we have

$$\begin{aligned} &\mathbb{E}\{D_1 + Vf | Q(t)\} \\ &\le \frac{1}{2}\mathbb{E}\{A_i(t)^2 + b_i(t)^2 - 2A_i(t)\tilde{b}_i(t) + 2Q_i(t)(A_i(t) - b_i(t)) | Q(t)\} \\ &\le B'^2 + \mathbb{E}\{Q_i(t)(A_i(t) - b_i(t)) | Q(t)\} \end{aligned}$$

Because QLA-E greedily minimizes $D_1 + Vf$, we have

$$\begin{aligned} &\mathbb{E}\{D_1 + Vf | Q(t)\} \\ &\le B'^2 + \mathbb{E}\{Q_i(t)(A_i(t)^{\mathrm{ALT}} - b_i(t)^{\mathrm{ALT}}) | Q(t)\} \end{aligned}$$

where ALT represents any other alternate policy.

Now using OQLA as ALT, using Lemma 2, we have

$$\begin{aligned} \mathbb{E}\{D_1 + Vf | Q(t)\} &\le B'^2 + g_0(Q(t), V) \\ &\le B'^2 + Vf^{\mathrm{opt}} \end{aligned}$$

Taking expectations over Q(t) and summing the above over $t = 1, \ldots, T-1$, we have:

$$\mathbb{E}\{L_1(T) - L_1(1)\} + V\sum_{\tau=1} T f(\tau) \le TB'^2 + VTf^{\mathrm{opt}}$$

Rearranging the terms, using the facts that $L(t) \ge 0$ and $L(0) = 0$, dividing both sides by VT, and taking the limsup as $T \to \infty$, we get:

$$\limsup_{T\to\infty} \bar{f} \le f^{\mathrm{opt}} + \frac{B'^2}{V}$$

Proof completes by letting $B_2 = B'^2$.

C. *Proof of Theorem 4*

Lemma 4. For the network under QLA-PDQ-P, the following two equations hold.

For any Q_1 and Q_2, we have

$$g_3(Q_1) \le g_3(Q_2) + (Q_1 - Q_2) \bullet (A(t) - b(t)) \quad (3)$$

For any Q, we have

$$\| Q^{s_i}(t+1) - \gamma_3^* \|^2 - \| Q(t) - \gamma_3^* \|^2 \le C_3 - 2(\gamma_3^* - Q(t)) \bullet (A(t) - b(t)) \quad (4)$$

Proof of Lemma 4 can be found in [11].

3) Property of Queue Backlog

From (3) and (4) we have for any Q

$$\| Q^{s_i}(t+1) - \gamma_3^* \|^2 - \| Q(t) - \gamma_3^* \|^2 \le C_2 - 2(g_{3,s_i}(\gamma_3^*) - g_{3,s_i}(Q(t)))$$

Using Lemma 1 and noting that g_3 is polyhedral it can be concluded that γ_3^* is the unique attraction point of queue backlog $Q(t)$. Noting that $g_1(Q) = g_3(Q - C)$. Thus $\gamma_3^* = \gamma_0^* + C$ if $\gamma_0^* \ge -C$.

4) Property of Utility Function

By [11], we have

$$\begin{aligned} &\mathbb{E}\{\Delta' + Vf|Q(t)\} \\ &\le \frac{1}{2}\mathbb{E}\{A_i(t)^2 + b_i(t)^2 - 2A_i(t)\tilde{b}_i(t) + \\ &2(Q_i(t) - C_i)(A_i(t) - b_i(t))|Q(t)\} \\ &\le B'^2 + \mathbb{E}\{(Q_i(t) - C_i)(A_i(t) - b_i(t))|Q(t)\} \end{aligned}$$

Because QLA-PDQ-P greedily minimizes $\Delta' + Vf$, we have

$$\begin{aligned} &\mathbb{E}\{\Delta' + Vf|Q(t)\} \\ &\le B'^2 + \mathbb{E}\{(Q_i(t) - C_i)(A_i(t)^{ALT} - b_i(t)^{ALT})|Q(t)\} \end{aligned}$$

where ALT denotes represents any other alternate policy. Now using OQLA as ALT, using Lemma 2, we have

$$\begin{aligned} &\mathbb{E}\{\Delta' + Vf|Q(t)\} \le B'^2 + g_0(Q(t) - C) \\ &\le B'^2 + Vf^{opt} \end{aligned}$$

Following the same line as in Section V-B, we have

$$\limsup_{T\to\infty} \bar{f} \le f^{opt} + \frac{B'^2}{V}$$

Proof completes by letting $B_3 = B'^2$.

D. Proof of Theorem 2

Using Theorem 4 while letting $C = \gamma_0^*$ completes the proof.

E. Proof of Theorem 5

1)Property of Queue Backlog

Following the same line as in Section V-C, the relationship between γ_2^* and γ_4^* can be obtained.

2)Property of Utility Function

Similar to the one in Section V-C. The difference is that QLA-PDQ-E greedily minimizes $D' + Vf$.

VI. Simulation

In this section we provide simulation results for the QLA-PDQ-P, QLA-PDQ-E and QLA-VPDQ on the network model in Section II-D. QLA-P and QLA-E are omitted here because they are specific form of QLA-PDQ-P and QLA-PDQ-E respectively. We simulate QLA-PDQ-P and QLA-PDQ-E with $V_i = 10 + 100i (i = 0, \dots, 9)$ and $C_j = -500 + 100j (j = 0, \dots, 8)$, where bold symbol means a vector with all components equaling the same constant. Precision of p_{ijk} and a_{ij} is set to 0.01. We run each case for 10^4 timeslots under both algorithms. Under each value of V and C, average queue backlog and utility function are obtained by using the final 5000 timeslots when the network is in the steady-state (Fig. 1 and 2).

Linear relationship between attraction point and $\mathbf{C}$ mentioned in Theorem 4 and 5 are shown in Fig. 1(b) and 2(b). Linear relationship between attraction point and V mentioned in Theorem 2 and 3 are shown in Fig. 1(a) and 2(a). From Fig. 1(b) and 2(b), Fig. 1(a) and 2(a) it can be seen that $\boldsymbol{\gamma}_2^*$ approximately equals $\boldsymbol{\gamma}_0^*$ as mentioned in Theorem 3.

It can be seen from Fig. 1(d) and 2(d) that f decreases dramatically if $\mathbf{C}$ decreases when $\mathbf{C} \le -\boldsymbol{\gamma}_0^*$. However, theoretical relationship between $\mathbf{C}$ and f when $\mathbf{C} \le -\boldsymbol{\gamma}_0^*$ remains an open question. This question may relate to the property of g_0. However, we can see from Fig. 1(c) and 2(c) that $f \to f^{opt}$ as $V \to \infty$ for all values of $\mathbf{C}$, as mentioned in Theorem 4 and 5.

VII. Conclusion

We have investigated several variants of OQLA in this paper. First, rationality of OQLA is proved for using the first-order part of upper bound of drift of a function. Although the entire expression of drift is not used in OQLA, backlog and utility of OQLA performance is the same. Although the Lyapunov function of the network is not used in OQLA, backlog of OQLA halved and utility performance is the same. Therefore it is of no need to use either the entire expression or the Lyapunov function of the network. Second, linear relationship between perturbation in data queues and attraction point of the backlog is found.

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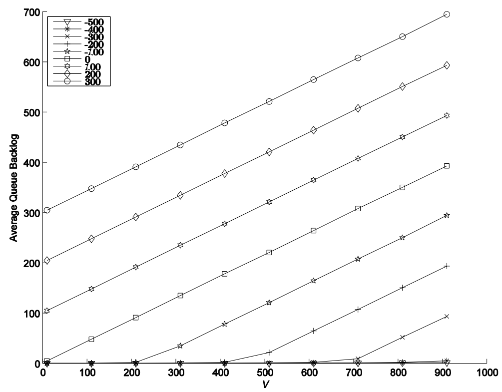
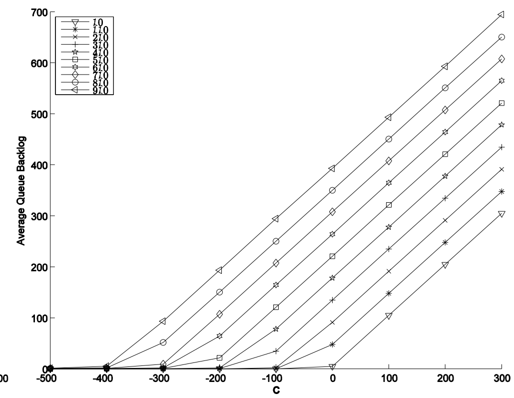
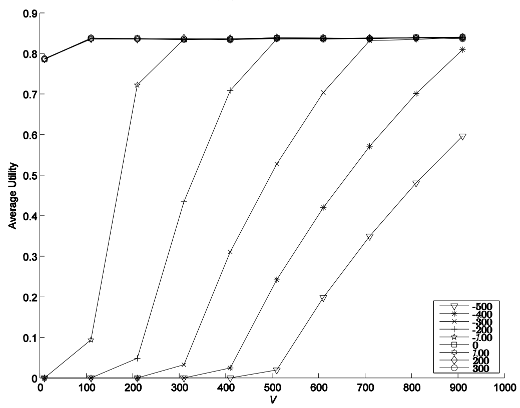


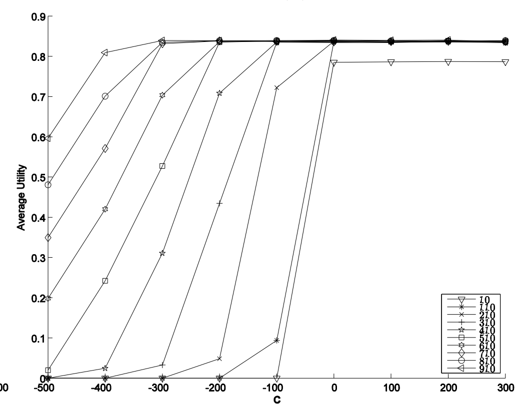
(a)



(b)

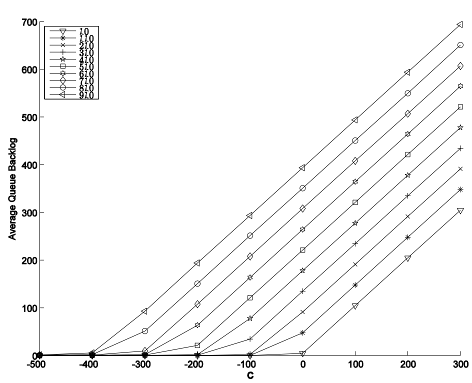


(c)

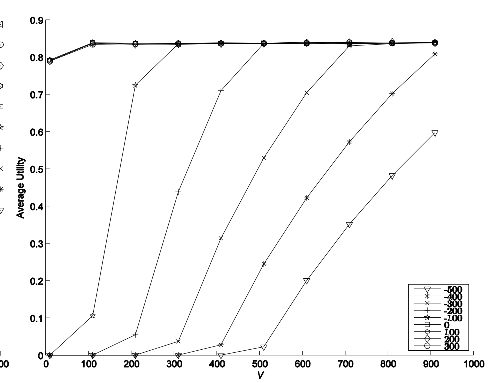


(d)

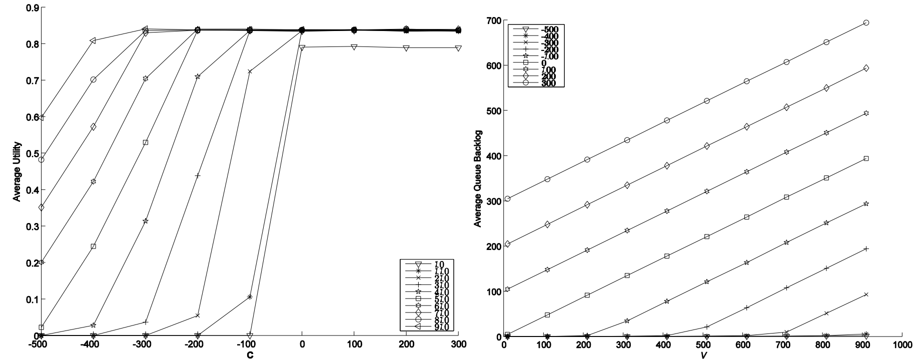
Figure 1. Simulation Results for QLA-PDQ-P: (a) Average Queue Backlog vs. V; (b) Average Queue Backlog vs. $\boldsymbol{C}$; (c) Average Utility vs. V; (d) Average Utility vs. $\boldsymbol{C}$



(a)



(b)



(c) (d)

Figure 2. Simulation Results for QLA-PDQ-E: (a) Average Queue Backlog vs. V ; (b) Average Queue Backlog vs. C; (c) Average Utility vs. V ; (d) Average Utility vs. C.